

Self-consistent anisotropic oscillator with cranked angular and vortex velocities

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Rotating deformed nuclei are neither rigid rotors nor irrotational droplets. The Kelvin circulation vector is the kinematical observable that measures the true character of nuclear rotation. For the anisotropic oscillator potential, mean field solutions with fixed angular momentum L and Kelvin circulation C are derived in analytic form. The cranking Lagrange multipliers corresponding to the L and C constraints are the angular ω and vortex λ velocities. Self-consistent solutions are reported with a constraint to constant volume.

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I. INTRODUCTION

The familiar adiabatic rotor model enables the determination of *static* nuclear shapes from collective multipole transition data. By applying the Alaga rules of this simple geometrical model, experimental $E2$ transition rates are interpreted in terms of the β and γ shape deformation parameters. The Riemann rotor model is a similar enabling model for *dynamical* nuclear currents. [1] Transverse form factors and other experimental measures of nuclear collective dynamics may be interpreted in terms of the rigidity parameter r . This parameter ranges continuously between the limiting cases of rigid rotation $r = 1$ and irrotational flow $r = 0$. The observable measuring the static deformation is the quadrupole operator $Q^{(2)}$; the vector observables measuring the dynamical current are the angular momentum \vec{L} and the Kelvin circulation \vec{C} .

The first aim of this paper is to show that the classical expressions [2,3] for the kinetic energy, angular momentum, and Kelvin circulation of a Riemann rotor may be derived by simultaneous angular and vortex “Inglis” cranking of the quantum anisotropic oscillator [4,5]

$$H_{\omega\lambda} = -\frac{\hbar^2}{2m} \Delta + \frac{1}{2}m(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2) - (\omega \hat{L}_x - \lambda \hat{C}_x). \quad (1)$$

This semiclassical correspondence is achieved in perturbation theory for small angular and vortex velocities provided the field is self-consistent with the shape,

$$\omega_x N_x = \omega_y N_y = \omega_z N_z, \quad (2)$$

where $N_k = \sum (n_k + 1/2)$ denotes the total number of quanta in the k^{th} direction. When the vortex velocity vanishes $\lambda = 0$, self-consistency implies rigid rotation, a well-known result for ordinary Inglis cranking of the

angular velocity. [6,7] When $\lambda \neq 0$, self-consistency implies the Riemann rotor moment of inertia which is an interpolation between the rigid and irrotational flow moments. Next, analytic formulas are discovered for the exact eigenvalues of the Routhian $H_{\omega\lambda}$. Calculations at finite ω and λ are reported for ^{20}Ne and ^{166}Er with a constraint to constant volume.

To define and interpret physically the vortex velocity and the Kelvin circulation, consider a system of A particles and its associated inertia ellipsoid. With respect to an inertial center-of-mass frame, the inertia ellipsoid rotates with angular velocity $\vec{\omega}$. By definition, the inertia tensor in the rotating intrinsic frame is diagonal,

$$Q_{ij} = \sum_{\alpha=1}^A x_{\alpha i} x_{\alpha j} = \frac{A}{5} a_i^2 \delta_{ij}, \quad (3)$$

where \vec{x}_α denotes the coordinates of particle α in the body-fixed frame and the a_i are the axes lengths of the inertia ellipsoid. In quantum mechanics, the angular momentum and circulation are defined by the operators [1]

$$\begin{aligned} \hat{L}_k &= -i\hbar \sum_{\alpha} \left(x_{\alpha i} \frac{\partial}{\partial x_{\alpha j}} - x_{\alpha j} \frac{\partial}{\partial x_{\alpha i}} \right) \\ \hat{C}_k &= -i\hbar \sum_{\alpha} \left(\frac{a_j}{a_i} x_{\alpha i} \frac{\partial}{\partial x_{\alpha j}} - \frac{a_i}{a_j} x_{\alpha j} \frac{\partial}{\partial x_{\alpha i}} \right) \\ &= -i\hbar \sum_{\alpha} \left(y_{\alpha i} \frac{\partial}{\partial y_{\alpha j}} - y_{\alpha j} \frac{\partial}{\partial y_{\alpha i}} \right), \end{aligned} \quad (4)$$

where $y_{\alpha i} = x_{\alpha i}/a_i$ are the dimensionless coordinates of particle α in the stretched intrinsic coordinate system and i, j, k are cyclic. In the stretched system, the inertia ellipsoid is transformed into a sphere of unit radius, and the inertia tensor is a multiple of the identity matrix. The Kelvin circulation is evidently the generator of rotations in this stretched frame. Adopting the active viewpoint, the operator $\exp(-i\vec{\omega} \cdot \vec{L}/\hbar)$ generates a finite rotation of the nuclear system with an angular velocity $\vec{\omega}$ with respect to the laboratory frame. The operator $\exp(i\vec{\lambda} \cdot \vec{C}/\hbar)$ generates a vortex rotation of the nucleons with respect to the body-fixed frame with the vortex velocity $\vec{\lambda}$.

In this introductory section, the theory of classical Riemann rotors is reviewed briefly. A classical Riemann rotor is a constant density fluid with an ellipsoidal boundary and a velocity field that is a linear function of the position coordinates. Classical Riemann rotors provide models for rotating stars and galaxies. [2,3] For nuclei, a linear velocity field was proposed by Cusson. [8] For these classical rotors, the uniform vorticity $\vec{\zeta}$, defined as the curl of the velocity field with respect to the body-fixed frame, is given by

$$\zeta_k = -\frac{a_i^2 + a_j^2}{a_i a_j} \lambda_k. \quad (5)$$

The curl of the laboratory frame velocity field \vec{U} resolved along the intrinsic axes is the inertial frame vorticity,

$$\vec{\zeta}^{(0)} = \vec{\nabla}_x \times \vec{U} = \vec{\zeta} + 2\vec{\omega}. \quad (6)$$

Then the Kelvin circulation of a Riemann rotor, defined by the line integral of the velocity field around the ellipse C_k bounding the fluid in the $i-j$ principal plane, is given via Stoke's theorem by the expression

$$C_k = \frac{M}{5\pi} \oint_{C_k} \vec{U} \cdot d\vec{l} = \frac{M}{5} a_i a_j (\zeta_k + 2\omega_k), \quad (7)$$

since $\pi a_i a_j$ is the area of the ellipse C_k . M is the fluid's mass. Note that irrotational flow is attained if the circulation \vec{C} and the inertial frame vorticity $\vec{\zeta}^{(0)}$ vanish, and the body-fixed uniform vorticity satisfies $\vec{\zeta} = -2\vec{\omega}$. For a classical Riemann rotor, the angular momentum and circulation are

$$\begin{aligned} L_k &= (M/5) [(a_i^2 + a_j^2)\omega_k - 2a_i a_j \lambda_k] \\ C_k &= (M/5) [2a_i a_j \omega_k - (a_i^2 + a_j^2)\lambda_k]. \end{aligned} \quad (8)$$

Ignoring vibrations of the axes lengths, the kinetic energy of a classical Riemann rotor is a combination of centrifugal and Coriolis terms

$$T(\vec{\omega}, \vec{\lambda}) = \frac{1}{2} (\vec{\omega} \cdot \vec{L} - \vec{\lambda} \cdot \vec{C}). \quad (9)$$

The angular momentum and circulation are given by derivatives of the kinetic energy with respect to the angular velocity and the vortex velocity

$$L_k = \left(\frac{\partial T}{\partial \omega_k} \right)_\lambda \quad C_k = - \left(\frac{\partial T}{\partial \lambda_k} \right)_\omega. \quad (10)$$

A Riemann ellipsoid in equilibrium has constant axes lengths, angular momentum and Kelvin circulation. The physical interpretation of equilibrium Riemann ellipsoids is most transparent for an S-type Riemann rotor that rotates about one principal axis, say the x -axis. In this case, the angular momentum, Kelvin circulation, angular velocity, and vortex velocity vectors are aligned with the x -axis and the kinetic energy simplifies to

$$T(\omega, \lambda) = \frac{I_0}{4} \{ (b^2 + c^2)(\omega^2 + \lambda^2) - 4bc\omega\lambda \}, \quad (11)$$

where $b = a_2/R$ and $c = a_3/R$ are the y and z axes lengths in units of a characteristic length R , and the moment of inertia of a sphere of radius R is $I_0 = (2/5)MR^2$.

The S-type equilibrium solutions are classified by a single parameter, the *rigidity* $r = 1 + \zeta/(2\omega)$. When $r = 1$ the vortex velocity is zero and the rotation is rigid. When $r = 0$ the circulation vanishes and the velocity field is irrotational, because the circulation is directly proportional to the rigidity and the cross-sectional area of the bounding ellipse,

$$C = I_0 b c r \omega. \quad (12)$$

Riemann rotors with $0 < r < 1$ span the full range of dynamical potentialities from irrotational flow to rigid rotation, consistent with the constraint to a S-type linear velocity field. The angular momentum, kinetic energy, and velocity field share a remarkable property for S-type ellipsoids: Each of these quantities is a simple convex combination of their corresponding rigid rotor (RR) and irrotational flow (IF) values,

$$L = I_r \omega \quad (13a)$$

$$T = \frac{1}{2} I_{r^2} \omega^2 \quad (13b)$$

$$U = r U_{\text{RR}} + (1 - r) U_{\text{IF}}, \quad (13c)$$

where the interpolated inertia is $I_p \equiv (p I_{\text{RR}} + (1 - p) I_{\text{IF}})$ for $0 \leq p \leq 1$ and the rigid body and irrotational flow moments of inertia are given by

$$I_{\text{RR}} = \frac{I_0}{2} (b^2 + c^2), \quad I_{\text{IF}} = \frac{I_0}{2} \frac{(c^2 - b^2)^2}{(c^2 + b^2)}. \quad (14)$$

In terms of the angular momentum, the kinetic energy is

$$T = \frac{L^2}{2 I_r}, \quad \mathcal{I}_r = \frac{(I_r)^2}{I_{r^2}}. \quad (15)$$

In the classical theory of fluid dynamics, the net Kelvin circulation is proved to be conserved by the hydrodynamic equations of motion. [2,3] Hence, to the extent that noncollective degrees of freedom may be ignored, geometrical states forming rotational bands in nuclei are conjectured to be approximate eigenstates of the Kelvin circulation.

A direct experimental method for determining the nuclear circulation and rigidity is provided by inelastic electron scattering measurements of the transverse $E2$ form factor, as has been emphasized by Moya de Guerra [9] and Vassanji and Rowe. [10] The author has shown elsewhere that the transverse $E2$ form factor for a Riemann rotor is a weighted interpolation of the rigid rotor and irrotational flow form factors

$$\mathcal{F}^{E2}(q) = [r I_{\text{RR}} \mathcal{F}_{\text{RR}}^{E2}(q) + (1 - r) I_{\text{IF}} \mathcal{F}_{\text{IF}}^{E2}(q)] / I_r, \quad (16)$$

where $\hbar q$ is the momentum transferred in the inelastic electron scattering. [11]

There is to date no published experimental measurement of transverse form factors in the heavy deformed region. However, a method based on measurement of multiple real photons in coincidence with the scattered electron was proposed recently that may permit the separation of the transverse from the longitudinal component due to interference terms in the angular distribution formulae [12]. If data were available, one would fit the expression for the Riemann form factor to the first peak and, thereby, measure the experimental rigidity r .

The projected Hartree-Fock calculations of the transverse form factor in the heavy deformed region made by

Berdichevsky *et al.* [13] enable a theoretical estimate for the rigidity. For ^{156}Gd , these PHF calculations correspond to a rigidity $r \sim 0.12$. This value compares favorably with the value predicted via Eq. (15) from the ^{156}Gd measured moment of inertia and deformation, $r \sim 0.15$.

This application of the Riemann model to dynamical currents is similar to the application of the adiabatic rotor model to static shapes. The physical meaning of a B(E2) transition rate, either measured experimentally or calculated in a detailed theoretical model, is obtained by its rotor model interpretation in terms of the nuclear shape, i.e., β and γ parameters. Similarly, the Riemann model attaches physical meaning to measurements and calculations of transverse $E2$ form factors in terms of the Kelvin circulation and the rigidity parameter r .

II. CRANKED ANISOTROPIC OSCILLATOR

A. Semiclassical Correspondence

To solve the energy eigenvalue problem for the cranked anisotropic oscillator Hamiltonian, introduce the oscillator creation and annihilation bosons and re-express the single-particle Kelvin circulation of Eq. (4) as

$$\begin{aligned} \hat{C}_x = \frac{-i\hbar}{2\sqrt{\omega_y\omega_z}} & \left\{ \left(\frac{c}{b}\omega_z + \frac{b}{c}\omega_y \right) (c_y^\dagger c_z - c_z^\dagger c_y) \right. \\ & \left. + \left(\frac{b}{c}\omega_y - \frac{c}{b}\omega_z \right) (c_y^\dagger c_z^\dagger - c_z c_y) \right\}. \end{aligned} \quad (17)$$

The circulation operator reduces to the angular momentum operator \hat{L}_x , if the axes lengths are replaced by unity. Substituting these expressions for the Kelvin circulation and angular momentum operators, the single-particle Routhian for the anisotropic oscillator potential is written in terms of bosons as

$$\begin{aligned} H_{\omega\lambda} = \hbar\omega_x(c_x^\dagger c_x + \frac{1}{2}) & + \hbar\omega_y(c_y^\dagger c_y + \frac{1}{2}) + \hbar\omega_z(c_z^\dagger c_z + \frac{1}{2}) \\ & - (\omega\hat{L}_x - \lambda\hat{C}_x). \end{aligned} \quad (18)$$

In perturbation theory, for small angular and vortex velocities, the collective kinetic energy $T(\omega, \lambda)$ of the A -nucleon system is given by Inglis's cranking formula: [4]

$$\begin{aligned} T = \sum_{ph} & \frac{|\langle p | \omega\hat{L}_x - \lambda\hat{C}_x | h \rangle|^2}{\epsilon_p - \epsilon_h} \\ = \frac{\hbar}{4\omega_y\omega_z} & \left\{ \left| \omega(\omega_y + \omega_z) - \lambda \left(\frac{b}{c}\omega_y + \frac{c}{b}\omega_z \right) \right|^2 \frac{N_z - N_y}{\omega_y - \omega_z} \right. \\ & \left. + \left| \omega(\omega_y - \omega_z) - \lambda \left(\frac{b}{c}\omega_y - \frac{c}{b}\omega_z \right) \right|^2 \frac{N_z + N_y}{\omega_y + \omega_z} \right\}. \end{aligned} \quad (19)$$

The second half of this equation is proven by applying the same techniques that work for the $\lambda = 0$ case. [5] Also, in

perturbation theory, the expectations of the axes lengths are given by

$$\frac{a^2}{5}R^2 = \frac{1}{A}\langle \sum_{\alpha=1}^A x_\alpha^2 \rangle = \frac{\hbar}{mA} \frac{N_x}{\omega_x}, \quad (20)$$

and similarly for b^2 and c^2 . Self-consistency of the shape of the potential field with the spatial density distribution requires equality for the ratios

$$a : b : c = \frac{1}{\omega_x} : \frac{1}{\omega_y} : \frac{1}{\omega_z}, \quad (21)$$

viz., Eq. (2). A principal result of this paper is the following semiclassical correspondence for the cranked anisotropic oscillator:

Theorem 1 *For self-consistent perturbation solutions to the cranked anisotropic oscillator $H_{\omega\lambda}$, the Inglis cranking energy, Eq. (19), equals the classical Riemann rotor energy, Eq. (11). The expectations of the angular momentum and the Kelvin circulation are given by their classical values too, Eq. (8), and satisfy the derivative conditions, Eq. (10).*

This theorem is proved by using the self-consistency relation and the formulae for the expectations of the axes lengths, Eq. (20), to eliminate the total number of quanta N_i and the frequencies ω_i from the perturbation expressions for the energy eigenvalue and for the expectations of the angular momentum and Kelvin circulation operators.

B. Analytic Quantum Mean-Field Results

The Routhian eigenvalue problem may be solved analytically by making a canonical transformation from the original oscillator bosons to new bosons that diagonalize $H_{\omega\lambda}$. This transformation exists because the Routhian is a quadratic form in the oscillator bosons. [14,15] The exact solution to the rigid rotor $\lambda = 0$ eigenvalue problem is known already. [16] For $\lambda \neq 0$, the eigenvalues of the A-particle Routhian are given by

$$\tilde{E} = \hbar\omega_x N_x + \hbar\Omega_+ N_y + \hbar\Omega_- N_z, \quad (22)$$

where the frequencies are

$$\Omega_\pm^2 = \frac{1}{2}(\omega_y^2 + \omega_z^2) + \omega^2 + \lambda^2 - \left(\frac{c}{b} + \frac{b}{c}\right)\omega\lambda \pm q \quad (23)$$

and

$$\begin{aligned} q^2 = & \frac{1}{4}(\omega_y^2 - \omega_z^2)^2 + (\omega_y^2 + \omega_z^2)(2\omega^2 + \lambda^2) \\ & + \left(\frac{b^2}{c^2}\omega_y^2 + \frac{c^2}{b^2}\omega_z^2\right)\lambda^2 \\ & - \left(\left(\frac{c}{b} + 3\frac{b}{c}\right)\omega_y^2 + \left(\frac{b}{c} + 3\frac{c}{b}\right)\omega_z^2\right)\omega\lambda. \end{aligned} \quad (24)$$

Note that in the limit of null cranking,

$$\lim_{\omega, \lambda \rightarrow 0} \Omega_{\pm} = \omega_z. \quad (25)$$

The expectations of the position operators, angular momentum, and circulation may be calculated by using Feynman's lemma that identifies the expectation of a derivative of the Hamiltonian operator to the corresponding derivative of the energy expectation. [17] For example, the expectation $\langle x^2 \rangle$ is determined by

$$\begin{aligned} \left\langle \frac{\partial H_{\omega\lambda}}{\partial \omega_x} \right\rangle &= \frac{\partial E}{\partial \omega_x} \\ m\omega_x \left\langle \sum_{\alpha=1}^A x_{\alpha}^2 \right\rangle &= \hbar N_x \\ \frac{I_0}{2\hbar} a^2 &= \frac{N_x}{\omega_x}. \end{aligned} \quad (26)$$

The expectations of the other position operators are determined *mutatis mutandis*:

$$\begin{aligned} \frac{I_0}{2\hbar} b^2 &= \frac{1}{2} \left(\frac{N_y}{\Omega_+} + \frac{N_z}{\Omega_-} \right) + \frac{1}{2q} \left(\frac{N_y}{\Omega_+} - \frac{N_z}{\Omega_-} \right) \left[\frac{1}{2} (\omega_y^2 - \omega_z^2) + 2\omega^2 + \left(1 + \frac{b^2}{c^2} \right) \lambda^2 - \left(\frac{c}{b} + 3\frac{b}{c} \right) \omega\lambda \right] \\ \frac{I_0}{2\hbar} c^2 &= \frac{1}{2} \left(\frac{N_y}{\Omega_+} + \frac{N_z}{\Omega_-} \right) - \frac{1}{2q} \left(\frac{N_y}{\Omega_+} - \frac{N_z}{\Omega_-} \right) \left[\frac{1}{2} (\omega_y^2 - \omega_z^2) - 2\omega^2 - \left(1 + \frac{c^2}{b^2} \right) \lambda^2 + \left(\frac{b}{c} + 3\frac{c}{b} \right) \omega\lambda \right]. \end{aligned} \quad (27)$$

The expectations of the angular momentum and circulation are calculated from derivatives of the energy with respect to ω and λ , Eq. (10),

$$\begin{aligned} \langle \hat{L}_x \rangle &= -\hbar \left(\frac{N_y}{\Omega_+} + \frac{N_z}{\Omega_-} \right) \left(\omega - \frac{1}{2} \left(\frac{c}{b} + \frac{b}{c} \right) \lambda \right) - \frac{\hbar}{2q} \left(\frac{N_y}{\Omega_+} - \frac{N_z}{\Omega_-} \right) \left[2(\omega_y^2 + \omega_z^2)\omega - \frac{1}{2} \left(\left(\frac{b}{c} + 3\frac{c}{b} \right) \omega_z^2 + \left(\frac{c}{b} + 3\frac{b}{c} \right) \omega_y^2 \right) \lambda \right] \\ \langle \hat{C}_x \rangle &= \hbar \left(\frac{N_y}{\Omega_+} + \frac{N_z}{\Omega_-} \right) \left(\lambda - \frac{1}{2} \left(\frac{c}{b} + \frac{b}{c} \right) \omega \right) + \frac{\hbar}{2q} \left(\frac{N_y}{\Omega_+} - \frac{N_z}{\Omega_-} \right) \left[(b^2 + c^2) \left(\frac{\omega_y^2}{c^2} + \frac{\omega_z^2}{b^2} \right) \lambda - \frac{1}{2} \left(\left(\frac{c}{b} + 3\frac{b}{c} \right) \omega_y^2 + \left(\frac{b}{c} + 3\frac{c}{b} \right) \omega_z^2 \right) \omega \right]. \end{aligned} \quad (28)$$

To leading order in ω and λ , these exact quantum results agree with the classical Riemann rotor formulas, as guaranteed by the theorem. In particular, the quantum collective energy is given by

$$\begin{aligned} E(\omega, \lambda) &= \hbar(\Omega_+ - \omega_y)N_y + \hbar(\Omega_- - \omega_z)N_z \\ &\quad + (\omega \langle \hat{L}_x \rangle - \lambda \langle \hat{C}_x \rangle), \end{aligned} \quad (29)$$

which equals $T(\omega, \lambda)$ to quadratic order in ω and λ .

C. Applications

Consider the case of ^{20}Ne for which $N_x = N_y = 14$, $N_z = 22$. If $\omega = \lambda = 0$, then the intrinsic energy is minimized, subject to a constraint to constant volume, when the self-consistency condition is satisfied, Eq. (2), or, equivalently, $\omega_i = \omega_0(N_x N_y N_z)^{1/3}/N_i$, where $\omega_0^3 = \omega_x \omega_y \omega_z$. Fixing $\hbar\omega_0 = 13.05$ MeV and $R = 3.257$ fm implies the following initial data: $\hbar\omega_x = \hbar\omega_y = 15.175$ MeV, $\hbar\omega_z = 9.656$ MeV, $a = b = 3.093$ fm, $c = 4.857$ fm. If ω or $\lambda \neq 0$, then the frequencies ω_i are chosen

so as to minimize the intrinsic energy in the rotating frame \tilde{E} , subject to the constraint of constant volume, i.e., the product of the axes lengths is fixed, $abc=46.465 \text{ fm}^3$. In addition, the axes lengths must be self-consistent with the definition of the Kelvin circulation, viz., equality is achieved in Eqs. (27). The results of the intrinsic energy minimization are plotted in Figures (1-5). The quantum angular momentum L and the quantum Kelvin circulation C are given by the semiclassical expressions $\langle L_x \rangle = \hbar \sqrt{L(L+1)}$ and $\langle C_x \rangle = \hbar \sqrt{C(C+1)}$. In Table I, self-consistent solutions that fit the ^{20}Ne experimental energy spectrum are tabulated. Note that the classical Riemann rotor kinetic energy $T(\omega, \lambda)$, Eq. (11), gives good agreement for small cranking velocities with the self-consistent quantum collective energy $E(\omega, \lambda)$, Eq. (29).

The most significant difference between the classical Riemann rotor formulas, valid for small cranking velocities ω and λ , and the self-consistent quantum results is that the quantum rotational band is cut off. This band termination is attained when the ellipsoid turns into an oblate spheroid rotating about its symmetry axis, $a < b = c$. At the cut off, $\omega_x > \omega_y = \omega_z$, $\Omega_{\pm} = \omega_y \pm |\omega - \lambda|$ and the maximal angular momentum and circulation are attained

$$\langle \hat{L}_x \rangle = \langle \hat{C}_x \rangle = \frac{\omega - \lambda}{|\omega - \lambda|} (N_z - N_y). \quad (30)$$

In the ^{20}Ne case, these maximal values are $N_z - N_y = 8$.

For the low energy states of a rotational band in a heavy deformed nucleus, the classical expressions are excellent approximations to the quantum cranking formulas. For example, in ^{166}Er the total number of deformed oscillator quanta are $N_x = N_y = 235$, $N_z = 343$. The band terminates when the angular momentum $L = N_z - N_y = 108$, and the classical results for a Riemann rotor are excellent approximations when $L, C \ll 108$. In Table II, self-consistent results for this heavy deformed nucleus are presented; the angular and vortex velocities are fit to the experimental energy and angular momentum. First, observe that the self-consistent collective energy $E(\omega, \lambda)$ is well approximated by the classical value $T(\omega, \lambda)$; the error in the classical formula is less than 0.03% up to angular momentum $L = 8$. Second, the axes lengths are constant up to $L = 8$. Third, the rigidity rises slowly with increasing angular momentum. Finally, the ratio of the Kelvin circulation to the angular momentum increases from 0.42 at $L = 2$ to 0.51 at $L = 8$.

III. CONCLUSION

In this article, the quantum Riemann rotor model was formulated as a cranked mean field theory. If the mean field is approximated by the deformed oscillator potential, then self-consistent solutions correspond to classical

Riemann rotors at small cranking velocities. This semiclassical correspondence provides a physical interpretation to the cranked angular ω and vortex λ velocities of the quantum mean field theory.

In previous work in nuclear physics concerning the Riemann rotor model, an algebraic model provided the framework for the quantum formulation of the Riemann classical model. What is the connection between the method of this article and the prior algebraic work?

The algebraic framework provides a unifying perspective for the adiabatic rotational model and the Riemann rotor model in both their classical and quantum mechanical forms. The two geometrical collective models are associated with two subalgebras of the symplectic $\text{Sp}(3, \mathbf{R})$ Lie algebra, known as $\text{ROT}(3)$ and $\text{GCM}(3)$, [18]

$$\text{SO}(3) \subset \text{ROT}(3) \subset \text{GCM}(3) \subset \text{Sp}(3, \mathbf{R}). \quad (31)$$

The rotational algebra $\text{ROT}(3)$ is spanned by the one-body quadrupole operator $Q^{(2)}$ plus the angular momentum algebra $\text{SO}(3)$. The general collective motion Lie algebra $\text{GCM}(3)$ is spanned by the full inertia tensor Q plus the general linear group $\text{Gl}(3, \mathbf{R})$.

The classical models corresponding to $\text{ROT}(3)$ and $\text{GCM}(3)$ are defined on the phase spaces formed by coadjoint orbits in the duals of the Lie algebras of $\text{ROT}(3)$ and $\text{GCM}(3)$. [19–21] The Hamiltonian dynamical systems on these coadjoint orbits are identical to the classical Euler rigid rotor model for $\text{ROT}(3)$ and to the Riemann-Chandrasekhar-Lebovitz virial equations of motion for $\text{GCM}(3)$. [1]

The quantum models corresponding to $\text{ROT}(3)$ and $\text{GCM}(3)$ are created by making a decomposition of the Fock space of antisymmetrized A-fermion states into irreducible unitary representations of these two algebras. These two decompositions are achieved explicitly by making a change of variables to collective and intrinsic coordinates. [22–25] The collective coordinates are defined on the orbits of the motion groups $\text{SO}(3)$ and $\text{Gl}(3, \mathbf{R})$ in \mathbf{R}^{3A} ; the intrinsic coordinates are a smooth transversal to the orbit manifolds. For $\text{ROT}(3)$, the irreducible representation spaces correspond to the well-known adiabatic rotational model. [26,27] The $\text{GCM}(3)$ decomposition into collective and intrinsic coordinates is less familiar, because the end result is a poor approximation to the physics of dynamical nuclear currents, i.e., the coupling between the collective and intrinsic coordinates of $\text{GCM}(3)$ is not weak. [28–30]

A quantitative measure of the goodness of $\text{ROT}(3)$ and $\text{GCM}(3)$ symmetry is obtained by their respective Casimir operators. Within a single irreducible representation, a Casimir operator is a multiple of the identity operator. However, if $\text{ROT}(3)$ or $\text{GCM}(3)$ symmetry is a poor approximation and collective nuclear states cannot be represented accurately as the product of collective and intrinsic wavefunctions, then their corresponding Casimir operators will not be constant among the states of a rotational band. There are two Casimirs of

ROT(3), $[Q^{(2)} \times Q^{(2)}]^{(0)} \propto \beta^2$ and $[Q^{(2)} \times Q^{(2)} \times Q^{(2)}]^{(0)} \propto \beta^3 \cos 3\gamma$, which measure the deformation β and triaxiality γ of the inertia ellipsoid. [31–34] The two ROT(3) Casimirs are approximately constant if the nuclear shape is approximately constant, a good first approximation to nuclear rotational bands. Hence, ROT(3) symmetry is useful for nuclear rotors, and the predictions of the adiabatic rotational model (Alaga rules for E2 transitions) are good first approximations to the experimental data.

The GCM(3) Casimir invariant $\hat{C}^2 = \vec{\hat{C}} \cdot \vec{\hat{C}}$ is the squared length of the Kelvin circulation vector. The connection of this Casimir with the Kelvin circulation was not appreciated until recently; [1] $\vec{\hat{C}}$ was referred to as the vortex momentum by the discoverers of the GCM(3) Casimir operator [29,35]. They established the following expression for it:

$$\hat{C}_k = \sum_{ij} \epsilon_{ijk} (\hat{Q}^{-1/2} \hat{N} \hat{Q}^{1/2})_{ij}. \quad (32)$$

In the intrinsic rotating frame, $\hat{Q} = \text{diag}(a^2, b^2, c^2)$ is diagonal and the $\text{Gl}(3, \mathbf{R})$ generator is $\hat{N}_{ij} = \sum_{\alpha=1}^A x_{\alpha i} p_{\alpha j}$. Thus, the above definition specializes to Eq. (4) in the intrinsic frame coordinates. In quantum mechanics, the net circulation is quantized to nonnegative integral multiples of \hbar and its squared length \hat{C}^2 to $C(C+1)\hbar^2$.

This article’s cranking calculations demonstrate that the GCM(3) Casimir is not even roughly constant among the states of nuclear rotational bands. Indeed, for a heavy deformed nucleus, it is the rigidity r and axes lengths a, b, c that are approximately constant; hence, the Kelvin circulation is approximately proportional to the angular velocity, Eq. (12). Therefore, GCM(3) symmetry is not found in real nuclei. In the algebraic approach, this defect is remedied by extending the dynamical group to $\text{Sp}(3, \mathbf{R})$. [18]

Because of the complexity of the Kelvin circulation operator, $\text{Sp}(3, \mathbf{R})$ shell model calculations to date have not attempted to determine the expectation of this operator with respect to microscopic wavefunctions. Heretofore, the only microscopic information available about the Kelvin circulation operator is a formal theorem that its eigenvalue spectrum within an infinite-dimensional irreducible symplectic shell model space is identical to the angular momentum spectrum of the associated $0\hbar\omega$ Elliott SU(3) representation. [30,36] For an axially symmetric $\text{Sp}(3, \mathbf{R})$ and SU(3) representation ($N_x = N_y$), the spectrum of C consists of the nonnegative integers from 0 to $N_z - N_y$. This agrees with the cutoff of this paper’s mean field theory.

The Kelvin circulation operator is intractable in a shell model theory because the square root and the inverse of the inertia tensor are part of its definition, Eq. (32). The inertia tensor is only diagonal in the intrinsic frame, a property that shell model theory cannot exploit. This paper’s mean field theory of Riemann rotors takes advantage of the intrinsic frame to simplify the Kelvin circulation operator by replacing the inertia tensor oper-

ator in the rotating frame by its c-number expectation, $\text{diag}(a^2, b^2, c^2)$. The inverse and square root are then trivial. This replacement is an approximation that ignores quantum shape fluctuations that are small for a deformed rotor compared to the other terms in the Kelvin circulation operator.

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FIG. 1. The collective energy $E(\omega, \lambda)$ of Eq. (29) is plotted in MeV versus the angular $\hbar\omega$ and vortex $\hbar\lambda$ velocities in MeV.

FIG. 2. The quantum angular momentum L in units of \hbar is plotted versus the angular $\hbar\omega$ and vortex $\hbar\lambda$ velocities in MeV. Here the quantum value $\sqrt{L(L+1)}$ is set equal to the semiclassical expectation $\langle \hat{L}_x \rangle$.

FIG. 3. The quantum Kelvin circulation C in units of \hbar is plotted versus the angular $\hbar\omega$ and vortex $\hbar\lambda$ velocities in MeV. Here the quantum value $\sqrt{C(C+1)}$ is set equal to the semiclassical expectation $\langle \hat{C}_x \rangle$.

FIG. 4. The expectation of the quadrupole operator $\langle Q_{20} \rangle = \langle 2z^2 - x^2 - y^2 \rangle$ in fm^2 is plotted versus the angular $\hbar\omega$ and vortex $\hbar\lambda$ velocities in MeV.

FIG. 5. The expectation of the quadrupole operator $\langle Q_{22} \rangle = \langle y^2 - x^2 \rangle$ in fm^2 is plotted versus the angular $\hbar\omega$ and vortex $\hbar\lambda$ velocities in MeV.

TABLE I. Self-consistent calculation for the yrast rotational band in ^{20}Ne

L	C	$\hbar\omega$ (MeV)	$\hbar\lambda$ (MeV)	E (keV)	T (keV)	r	a (fm)	b (fm)	c (fm)
0	0.0	0.0	0.0	0.0	0.0	-	3.093	3.093	4.857
2	0.921	2.170	1.539	1.634	1.640	.217	3.090	3.093	4.857
4	2.536	3.143	1.796	4.247	4.541	.371	3.090	3.103	4.841
6	4.120	4.592	2.383	8.775	10.527	.435	3.084	3.148	4.780

TABLE II. Self-consistent calculation for the yrast rotational band in ^{166}Er

L	C	$\hbar\omega$ (MeV)	$\hbar\lambda$ (MeV)	E (keV)	T (keV)	r	a (fm)	b (fm)	c (fm)
0	0.0	0.0	0.0	0.0	0.0	-	5.927	5.927	8.651
2	0.847	0.112	0.090	80.57	80.57	0.136	5.927	5.927	8.651
4	1.858	0.202	0.162	264.98	264.99	0.139	5.927	5.927	8.651
6	2.924	0.289	0.231	545.44	545.50	0.143	5.927	5.927	8.651
8	4.043	0.372	0.296	911.18	911.38	0.148	5.927	5.927	8.651